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Abstract. Jacobi brackets (a generalization of standard Poisson brackets in which Leibniz's rule is replaced by a weaker condition) are extended to brackets involving an arbitrary (even) number of functions. This new structure includes, as a particular case, the recently introduced generalized Poisson structures. The linear case on simple group manifolds is also studied and non-trivial examples (different from those coming from generalized Poisson structures) of this new construction are found by using the cohomology ring of the given group.

1. Introduction

Poisson structures (and Hamiltonian systems) can be introduced in geometrical terms by means of an appropriate bivector field Λ verifying certain compatibility conditions that can be formulated by imposing the vanishing of the Schouten–Nijenhuis bracket (SNB) [1, 2] of Λ with itself, $[\Lambda, \Lambda] = 0$ [3]. This construction neither makes reference to symplectic structures nor requires a manifold of even dimension and provides a very convenient approach to generalize standard Poisson brackets. Following this path, a generalization of standard Poisson structures has been introduced [4] based on even multivector fields $\Lambda \in \wedge^{(2p)}$ having zero SNB with themselves $[\Lambda, \Lambda] = 0$. In the linear case, this new *generalized Poisson structure* (GPS) admits an infinity of examples related to the higher-order Lie algebras [5], a fact which generalizes the well known isomorphism between linear Poisson structures constructed out of the structure constants and (ordinary) Lie algebras. The GPS are different from those proposed by Nambu long ago [6] where a (Nambu–)Poisson bracket involving three functions was introduced. Later Takhtajan [7] extended the Nambu construction to a Nambu–Poisson bracket with an arbitrary number of functions (see also [8–10]).

In this paper we construct a higher-order generalization of the Jacobi structures [11, 12], themselves a generalization of the standard Poisson structures, called local Lie algebras by Kirillov [13]. The generalization of the Poisson structures provided by the Jacobi ones is the result of substituting the Leibniz rule (derivation property) of the Poisson bracket by the weaker condition

$$\mathrm{support}\{f, g\} \subseteq \mathrm{support}\ f \cap \mathrm{support}\ g. \tag{1}$$

Then, it is possible to show [13] that the new bracket (*Jacobi bracket*) is a local type operator which has to be given by linear differential operators. This implies that Jacobi structures, in

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contrast with standard Poisson structures which may be determined uniquely by a bivector field Λ, are characterized by the differential operators defining the Jacobi bracket, namely a bivector and vector fields Λ and E. If we now want the new bracket to satisfy the (standard) Jacobi identity (see (3) below), Λ and E must verify some compatibility conditions that can be expressed in terms of the Schouten–Nijenhuis bracket [11, 12]. It is clear that *all* Poisson structures are also Jacobi structures because the Leibniz rule implies condition (1); this is the case when the vector field E is set equal to 0.

The aim of this paper is to show that, using the same geometrical approach by means of which (standard) Poisson structures can be extended to higher-order GPS, Jacobi structures can also be extended to higher-order *generalized Jacobi structures* (GJS). In these, the generalized Jacobi brackets involve an arbitrary even number of functions. They satisfy the same generalized Jacobi identity (GJI) introduced in [4] (see (17)) by virtue of which both linear differential operators (a $2p$ vector and a $(2p-1)$ vector field) defining the generalized Jacobi bracket are constrained by some conditions expressed by means of the SNB. When the $(2p-1)$ vector field is set equal to zero we recover a standard Poisson structure (for $p=1$) or a GPS (p arbitrary). As a result, all GPS are also generalized Jacobi structures. Although I have not been able to find a direct application of the GJS (which, as far as I know, is not easy even for the standard Jacobi structures), I have been able to provide an infinite number of examples of these structures in the linear case, which extends greatly their mathematical interest.

The paper is organized as follows. In section 2 the definition of Jacobi bracket and Jacobi manifold is recalled [11–13]. In section 3 the GJS are introduced and some examples given. Some conclusions close the paper.

2. Jacobi manifolds

Let $\mathcal{F}(M)$ be the associative algebra of functions on the manifold M.

Definition 2.1 (Jacobi bracket). A *Jacobi bracket* is a bilinear operation $\{,\}:\mathcal{F}(M)\otimes\mathcal{F}(M)\to\mathcal{F}(M)$ which satisfies (1) and the following conditions $\forall f,g,h\in\mathcal{F}(M)$:

(a) skew-symmetry

$$\{f,g\}=-\{g,f\} \tag{2}$$

(b) the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0. \tag{3}$$

Conditions (a) and (b) endow $\mathcal{F}(M)$ with a structure of Lie algebra. A manifold M with a Jacobi bracket is called a Jacobi manifold. If we substitute (1) for the stronger condition

$$\{f,gh\}=g\{f,h\}+\{f,g\}h \tag{4}$$

(Leibniz rule), we obtain a Poisson bracket (and then M is called a Poisson manifold).

The more general form of a Jacobi bracket on the manifold M is given [13] by

$$\{f,g\}=\Lambda(\mathrm{d}f,\mathrm{d}g)+fE(\mathrm{d}g)-gE(\mathrm{d}f) \tag{5}$$

where Λ and E are, respectively, a 2-vector and a vector field locally written as

$$\Lambda=\tfrac{1}{2}\Lambda^{ij}\partial_i\wedge\partial_j \qquad E=\xi^i\partial_i. \tag{6}$$

Condition (a) is automatically satisfied if $\{\,,\}$ is defined by (5). Condition (b) is taken into account by requiring

$$[\Lambda,\Lambda]=2E\wedge\Lambda \qquad [E,\Lambda]=0 \tag{7}$$

where [,] stands for the SNB [1, 2]. In fact (see [11])

$$\epsilon_{123}^{ijk}\{f_i, \{f_j, f_k\}\} = ([\Lambda, \Lambda] - 2E \wedge \Lambda)(\mathrm{d}f_1, \mathrm{d}f_2, \mathrm{d}f_3) - \epsilon_{123}^{ijk} f_i [E, \Lambda](\mathrm{d}f_j, \mathrm{d}f_k) \tag{8}$$

so that, by requiring (7), the Jacobi identity is satisfied. Thus [11], a Jacobi structure on M is defined by a 2-tensor Λ and a vector E satisfying the conditions (7).

It is clear that for $E = 0$ we recover the equation

$$[\Lambda, \Lambda] = 0 \tag{9}$$

which states that Λ is a Poisson bivector and that { , } defines a Poisson structure [3] on M.

In the same way that it is possible to characterize non-degenerate Poisson structures by covariant tensors satisfying $\mathrm{d}F = 0$, the Jacobi structures on a manifold of dimension $2n$ with non-degenerate bivector Λ are characterized [13, 12] by a 2-form F and a 1-form η which verify $\mathrm{d}F = \eta \wedge F$, where F and η are given by their coordinates defined by

$$\Lambda^{ik} F_{jk} = \delta^i_j \qquad \eta_i = F_{jk}\xi^k. \tag{10}$$

Examples of Jacobi structures (and Jacobi manifolds) are given by the locally conformal symplectic manifolds [14] defined on an even-dimensional manifold M through a non-degenerate 2-form Ω and a closed 1-form ω (the Lee form [15]) satisfying

$$\mathrm{d}\Omega = \omega \wedge \Omega \tag{11}$$

and the contact manifolds where we have a manifold M with $\dim M = 2n + 1$ and a 1-form ω on M (the contact form) which verifies

$$\omega \wedge (\mathrm{d}\omega)^n \neq 0 \qquad \forall x \in M. \tag{12}$$

We want to recall here the linear case.

Example 2.1. Let Ω be the Poisson bivector associated with a Poisson–Lie structure (i.e., $\Omega = \frac{1}{2}x_k C^k_{ij}\partial^i \wedge \partial^j$, where C^k_{ij} are the structure constants of a Lie algebra $\mathcal{G}$); then $[\Omega, \Omega] = 0$. If we define the dilatation vector field $A = x_i\partial^i$, we may check that $[A, \Omega] = -\Omega$. So, defining $\Lambda \equiv \Omega + E \wedge A$ and imposing $[\Lambda, E] = 0$ or, equivalently, $[E, \Omega] = -E \wedge [E, A]$ we obtain

$$[\Lambda, \Lambda] = [E \wedge A, E \wedge A] + 2[\Omega, E \wedge A] = 2E \wedge \Omega = 2E \wedge \Lambda. \tag{13}$$

Hence, the pair $(\Lambda \equiv \Omega + E \wedge A, E)$ defines a Jacobi structure if $[E, \Omega] = -E \wedge [E, A]$.

In particular, if E is a constant vector, the condition above is equivalent to the one-cocycle condition for E, which reads

$$\xi_\nu C^\nu_{ij} = 0. \tag{14}$$

For instance, if $\mathcal{G}$ is a simple (or semisimple) algebra the first cohomology group $H_1(\mathcal{G})$ is zero (Whitehead's lemma), but we can take the algebra $\mathcal{G} \otimes u(1)$ for which $H_1(\mathcal{G} \otimes u(1)) \neq 0$. Then, the bivector Λ is given by

$$\Lambda = \tfrac{1}{2}x_k C^k_{ij}\partial^i \wedge \partial^j + x_i\partial^\varphi \wedge \partial^i \tag{15}$$

where φ denotes the coordinate corresponding to the $u(1)$ algebra generator (see [16]).

3. Generalized Jacobi structures

A natural higher-order generalization of the standard Jacobi structures of definition 2.1 is given by $2p$ and $(2p-1)$ vector fields defining the linear mapping (cf (5))

$$\{f_1, \ldots, f_{2p}\} = \Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_{2p}) - \sum_{j=1}^{2p} (-)^j f_j E(\mathrm{d}f_1, \ldots, \widehat{\mathrm{d}f_j}, \ldots, \mathrm{d}f_{2p}) \tag{16}$$

which is antisymmetric in all its arguments f_i. Then, to define generalized Jacobi structures we still have to impose a generalized Jacobi identity. This leads to

Definition 3.1 (Generalized Jacobi structure). A *generalized Jacobi structure* on the manifold M is defined by a $2p$ and $(2p-1)$ vector fields (Λ, E) such that the mapping $\{\cdot, \ldots, \cdot\} : \mathcal{F}(M) \times \overset{2p}{\cdots} \times \mathcal{F}(M) \to \mathcal{F}(M)$ given by (16) satisfies the *generalized Jacobi identity* [4]

$$\epsilon^{j_1 \ldots j_{4p-1}}_{1 \ldots 4p-1} \{f_{j_1}, \ldots, f_{j_{2p-1}}, \{f_{j_{2p}}, \ldots, f_{j_{4p-1}}\} = 0 \qquad \forall f_j \in \mathcal{F}(M). \tag{17}$$

The bracket (16) will be called a *generalized Jacobi bracket*.

Now we need to characterize the generalized Jacobi structures in terms of the $2p$ and the $(2p-1)$ vector fields (Λ, E). This is achieved by the following.

Lemma 3.1 (Characterization of a GJS). The linear mapping (16) is a generalized Jacobi bracket (i.e., verifies (17)) *iff* Λ and E, written in a local chart (cf (6)) as

$$\Lambda = \frac{1}{2p!} \Lambda^{i_1 \ldots i_{2p}} \partial_{i_1} \wedge \cdots \wedge \partial_{i_{2p}} \qquad E = \frac{1}{(2p-1)!} \xi^{i_1 \ldots i_{2p-1}} \partial_{i_1} \wedge \cdots \wedge \partial_{i_{2p-1}} \tag{18}$$

satisfy

$$[\Lambda, \Lambda] = 2(2p-1) E \wedge \Lambda \qquad [E, \Lambda] = 0. \tag{19}$$

Proof. The structure of the proof is equivalent to that for the standard $p = 1$ case. In it we write the generalized Jacobi identity and factorize different kinds of terms. First we consider terms with first derivatives in f's. Those in (17) with the form $\partial f_1 \ldots \partial f_{4p-1}$ (all f's derived once) are proportional to $((2p-1)(E \wedge \Lambda) - \frac{1}{2}[\Lambda, \Lambda])$. Those with a non-derived f are either proportional to $E \wedge E$ and hence directly zero (E is of odd order) or proportional to $[E, \Lambda]$. Those with two non-derived f's (f_i, f_j say) are zero because they are symmetric under the permutation $f_i \leftrightarrow f_j$ while being antisymmetric the GJI in all the f's.

The terms with second derivatives are proportional to

$$\epsilon_{i_1 \ldots i_{4p-3}} (\Lambda^{i_1 \ldots i_{2p-1} \alpha} \xi^{\beta i_{2p} \ldots i_{4p-3}} + \xi^{i_1 \ldots i_{2p-2} \beta} \Lambda^{\alpha i_{2p-1} \ldots i_{4p-3}})$$

or to

$$\epsilon_{i_1 \ldots i_{n-1}\, j_1 \ldots j_{n-1}} (\Lambda^{i_1 \ldots i_{n-1} \alpha} \Lambda^{j_1 \ldots j_{n-1} \beta} + \Lambda^{i_1 \ldots i_{n-1} \alpha} \Lambda^{j_1 \ldots j_{n-1} \beta})$$

which are zero being E and Λ of odd and even order respectively. Thus, the unique conditions required to cancel all terms in the GJI are given by (19). $\square$

Corollary 3.1. In the particular case $E = 0$, (16) reduces to $\{f_1, \ldots, f_{2p}\} = \Lambda(\mathrm{d}f_1, \ldots, \mathrm{d}f_{2p})$ and (19) reduces to $[\Lambda, \Lambda] = 0$, i.e., Λ defines a GPS [4].

Example 3.1. Let M be a manifold with $\dim M > 2$; then if we take as Λ a $(\dim M)$ multivector field, for each $(\dim M - 1)$ vector E we have a pair (Λ, E) defining a GJS and M becomes a generalized Jacobi manifold. The conditions (19) are satisfied because $[\Lambda, \Lambda]$ and $E \wedge \Lambda$ are $(2\dim M - 1)$ vectors and $[\Lambda, E]$ is a $(2\dim M - 2)$ vector which are trivially zero on M.

This is a very simple example that, in some sense, generalizes the fact that a 2-vector on a two-dimensional manifold defines a (standard) Poisson structure.

Example 3.2. We can extend the linear example given in section 2 to this case. To this aim let Ω be a $2p$ vector field defining a linear generalized Poisson structure (see [4]), locally written as

$$\Omega = \frac{1}{2p!}\omega^k_{i_1\ldots i_{2p}} x_k \partial^{i_1} \wedge \cdots \wedge \partial^{i_{2p}} \tag{20}$$

and let A be the dilatation operator as in example 2.1. Then, for every $(2p - 1)$ vector field E satisfying $[E, \Omega] = -E \wedge [E, A]$ (that is, $[E, \Omega + E \wedge A] = 0$) we can define a generalized Jacobi structure given by the pair $(\Lambda \equiv \Omega + E \wedge A, E)$. In particular, if $E = (1/(2p-1)!)\xi_{i_1\ldots i_{2p-1}} \partial^{i_1} \wedge \cdots \wedge \partial^{i_{2p-1}}$ is a constant vector the condition on E reduces to the expression

$$\epsilon^{i_1\ldots i_{2p-2} j_1 \ldots j_{2p}}_{k_1\ldots k_{4p-2}} \xi_{\nu i_1\ldots i_{2p-2}} \omega^{\nu}_{j_1\ldots j_{2p}} = 0 \tag{21}$$

or, equivalently,

$$\partial_\Omega E = 0 \tag{22}$$

where ∂_Ω is the coboundary operator for the generalized Poisson cohomology introduced in [4]. In contrast with the standard $p = 1$ case, we do not need to 'extend' the algebra to find $(2p - 1)$-cocycles for the coboundary operator ∂_Ω†. In fact, as shown in [4] (see also [5, 17]), all the higher-order $\mathcal{G}$-cocycles for the ordinary Lie algebra cohomology are cocycles for the ∂_Ω cohomology. In other words, it is sufficient to find a simple Lie algebra with cocycles of orders $(2p - 1)$ and $(2p + 1)$ (or, in terms of the associated invariant polynomials, Casimirs of orders p and $p + 1$). This is the case, for instance, for $su(3)$ where we find the generalized Jacobi structure given by the pair $(\Omega + E \wedge A, E)$ where

$$\begin{aligned} \Omega &= \frac{1}{4!}\epsilon^{j_2 j_3 j_4}_{i_2 i_3 i_4} d^{\sigma}_{k_1 k_2} C^{k_1}_{i_1 j_2} C^{k_2}_{j_3 j_4} x_\sigma \partial^{i_1} \wedge \partial^{i_2} \wedge \partial^{i_3} \wedge \partial^{i_4} \\ E &= \frac{1}{3!} C_{i_1 i_2 i_3} \partial^{i_1} \wedge \partial^{i_2} \wedge \partial^{i_3} \end{aligned} \tag{23}$$

the coordinates $\xi_{ijk} = C_{ijk}$ of E are the structure constants of $su(3)$ and the d_{ijk} are the constants which appear in the anticommutators of the Gell–Mann matrices λ,

$$\{\lambda_i, \lambda_j\} = \tfrac{4}{3}\delta_{ij} 1_3 + 2 d_{ijk} \lambda_k. \tag{24}$$

The same construction extends to $su(l + 1) \sim A_l$ $(l \geqslant 2)$ for which we have l primitive invariant polynomials of orders $2, 3, \ldots, l+1$ and hence l cocycles of orders $3, 5, \ldots, 2l+1$. Thus, for every cocycle (different from the first one of order three which defines the standard Poisson/Jacobi structure) we can give a non-trivial generalized Jacobi structure. This explains why the standard case is singular and we have no linear Jacobi structures on the simple groups (defined by the tree-cocycle given by the structure constants which always exists).

† This is an important difference with the standard $p = 1$ case (section 2) in which we cannot define linear Jacobi structures on the dual of a simple Lie algebra.

4. Conclusions

Despite the lack of a Leibniz rule that permits us to define a simple dynamics by $\dot{f} = \{H, f\}$ (where $\{\,,\}$ stands for a Jacobi bracket) or, in the generalized case, $\dot{f} = \{H_1, \ldots, H_{2p-1}, f\}$ (see [4] for a discussion on generalized Poisson dynamics) the Jacobi structures are not devoid of physical (and mathematical) interest.

Generalized Poisson structures [4] (see also [18] for the Z_2-graded case) and their higher-order algebra counterparts [5] provide a particular example of *strongly homotopy algebras* [19, 20] which are relevant in certain structures appearing in closed string theory and in connection with the Batalin–Vilkovisky formalism (see e.g., [21, 22]; for an account of the Batalin–Vilkovisky formalism see [23, 24]). It has been mentioned recently [25] that there is a relation between Batalin–Vilkovisky algebras and Jacobi manifolds, although such a connection has not yet been made explicitly. Clearly, the standard and the generalized Poisson structures [4] are also special examples of the Jacobi structures considered here (it is sufficient to set $E = 0$ and add the Leibniz rule) and, as such, they may share some properties, but more work is needed to analyse any physical applications of the GJS and, in particular, their possible quantization. Note already that although (standard) Poisson brackets may be quantized by the bracket of associative operators that verifies the Leibniz rule

$$[A, BC] = ABC - BCA = [A, B]C + B[A, C]$$

(as well as skewsymmetry and Jacobi identity) the standard Jacobi structure does not satisfy this relation (unless it also defines a standard Poisson structure). Moreover, in general, the skewsymmetrized product of an arbitrary (even) number of associative operators does not satisfy the Leibniz rule (despite the fact that it verifies the generalized Jacobi identity [5]).

From a purely mathematical (but nevertheless relevant) point of view, the mathematical contents (see example 3.2) give to the new GJS a special interest, particularly in the linear case, where we have been able to provide examples associated with the cohomological properties of the Lie algebras. This raises the question of whether other relations among the cocycles of a given Lie algebra may give rise to generalized Jacobi brackets. This is matter for further work.

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